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ON THE REGULARITY OF THE PRINCIPAL VALUE OF THE DOUBLE LAYER POTENTIAL

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On the regularity of the principal value of the double layer potential *)

by

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ABSTRACT

In the present paper we discuss the regularity of the principal value of the potential due to a doublet distribution μ along the boundary S of a two-dimensional (2-D) open connected set. Assuming S to be a Lyapunov boundary and μ to be bounded and integrable we prove that the principal value in 2-D is more regular than the one in 3-D. This result is applied to the aerodynamic problem of calculation of potential flows around 2-D bodies.

KEY WORDS & PHRASES: *Double layer potential, regularity of the principal value, potential flow around aerofoils, Fredholm integral equation of the second kind, collocation method*

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*) This report will be submitted for publication elsewhere.

I. INTRODUCTION

The solution of the Dirichlet problem for Laplace's equation in two and three dimensions can be represented as a double layer potential, respectively:

$$(1.1.1) \qquad \phi_d^{(2)}(\zeta) = \frac{1}{\pi} \int_S \mu(z) \frac{\partial}{\partial n_z} \log|r_{z\zeta}| \cdot dS_z, \quad \zeta \notin S,$$

$$(1.1.2) \qquad \phi_d^{(3)}(\zeta) = \frac{1}{2\pi} \int_S \mu(z) \frac{\partial}{\partial n_z} 1/|r_{z\zeta}|\, dS_z, \quad \zeta \notin S,$$

where n_z is the outward normal to the surface S at the point z, $r_{z\zeta} = \zeta - z$ and $\mu(\cdot)$ is called the doublet distribution. These potentials are discontinuous across the surface. We denote the principal value of $\phi_d(\zeta)$ by

$$(1.2) \qquad \overline{\phi}_d^{(m)}(\zeta) = \frac{-2^{2-m}}{\pi} \int_S \mu(z) \frac{\cos(r_{z\zeta}, n_z)}{|r_{z\zeta}|^{m-1}} dS_z, \quad \zeta \in S,$$

where m = 2,3 for the two- and three-dimensional case, respectively. Assuming S to be a Lyapunov surface and μ to be bounded and integrable GÜNTER [1,p.49] proves that $\overline{\phi}_d^{(3)} \in H^{0,\alpha}(S)$, where $H^{k,\alpha}(S)$ denotes the class of continuous functions whose derivatives of order k satisfy a uniform Hölder condition with exponent α. In section 2 of the present paper we prove that in the two-dimensional case $\overline{\phi}_d^{(2)} \in H^{1,\alpha}(S)$, i.e. the principal value is more regular than in the three-dimensional case. This property of regularity has an important application in aerodynamics: the calculation of potential flow around aerofoils.

In this problem the doublet distribution μ is the solution of the following Fredholm equation of the second kind:

$$(1.3) \qquad \mu(\zeta) + \frac{1}{\pi} \int_S \mu(z) \frac{\cos(n_z, z-\zeta)}{|z-\zeta|} dS_z = -2U\cdot\zeta, \quad \zeta \in S,$$

where U is the velocity of the undisturbed flow. We write this equation in operator notation as

$$(1.4) \qquad (I-K)\mu = g,$$

with $K\mu(\zeta) = \overline{\phi}_d^{(2)}(\zeta)$ and $g(\zeta) = -2\mathcal{U}\cdot\zeta$. The result of the present paper shows that the linear integral operator K maps from the Banach space $L_\infty(S)$ of essentially bounded functions into the Banach space $H^{1,\alpha}(S)$. Moreover, the operator K is a bounded mapping from $L_\infty(S)$ into $H^{1,\alpha}(S)$.

In section 3 we shall approximate μ by a piecewise constant function μ_N. From the results of section 2 it follows that $K\mu_N \in H^{1,\alpha}(S)$. We shall use this result to supply error bounds for $\|\mu-\mu_N\|$ and $\max_i|\mu(\zeta_i) - \mu_i|$, where $\zeta_1,\zeta_2,\ldots,\zeta_N$ are the collocation points. This collocation method yields a large linear system of equations. In [6] we have iteratively solved this system by a multiple grid method. Using $K\mu_N \in H^{1,\alpha}(S)$ we were able to estimate the rate of convergence of the multiple grid process.

2. REGULARITY RESULT

First, we give some definitions which have been taken from GÜNTER [1]. Let $D \subset \mathbb{R}^2$ be an open connected set with boundary S and closure $\overline{D}$.

DEFINITION 2.1. $C^k(D)(C^k(\overline{D}))$ denotes the class of functions, which are k times continuously differentiable in $D(\overline{D})$.

DEFINITION 2.2. $C^{k,\alpha}(D)(C^{k,\alpha}(\overline{D}))$ denotes the subclass of functions in $C^k(D)(C^k(\overline{D}))$, whose derivatives of order k satisfy a uniform Hölder condition with exponent α, $0 < \alpha < 1$.

DEFINITION 2.3. $L^{k,\alpha}$ $(k \geq 1)$ denotes the class of boundaries S in 2-dimensional Euclidean space with the property that for every point P on S there exists a number $\varepsilon > 0$ such that the part Σ of S within the circle $B_{\varepsilon,P}$ of radius ε and centre P, for some orientation of the axes of the coordinate system (x,y), admits a representation

$$(2.1) \qquad y = F(x), \quad x \in \overline{D}_{\varepsilon,P},$$

where $F \in C^{k,\alpha}(\overline{D}_{\varepsilon,P})$, $\overline{D}_{\varepsilon,P}$ the projection of the part of S within $B_{\varepsilon,P}$ on the line $y = 0$.

We give an illustration of definition 2.3 in the following figure:

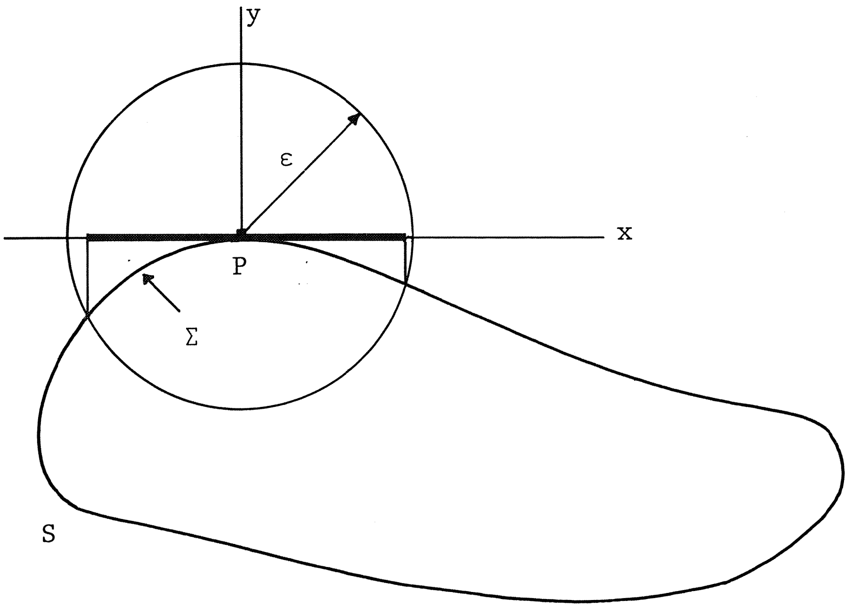

figure 2.1 Illustration of definition 2.3.
($\overline{D}_{\varepsilon,P}$ is given by ▬▬▬)

<u>DEFINITION 2.4</u>. $H^{k,\alpha}(S)$ denotes the class of functions f defined on S with the property that the function $\hat{f}$ defined by

$$\hat{f}(x) = f(x,F(x)), \quad x \in \overline{D}_{\varepsilon,P} ,$$

with $F(x)$ and $\overline{D}_{\varepsilon,P}$ as in definition 2.3, belongs to the class $C^{k,\alpha}(\overline{D}_{\varepsilon,P})$.

<u>REMARK 2.1</u>. Let $S \in L^{k,\alpha}$ with $k \geq 2$ and let $z,\zeta \in S$. Then

$$\lim_{\zeta \to z} \frac{2\cos(n_z, z-\zeta)}{|z-\zeta|} = \kappa(z),$$

where $\kappa(z)$ is the curvature at z. Moreover, κ belongs to the space $H^{k-2,\alpha}(S)$.

PROOF. Let (ξ,η) be a local coordinate system about a certain point $P \in S$ (see figure 2.1). By definition 2.3 the points z and ζ may be represented by $(x,F(x))$ and $(\xi,F(\xi))$, respectively. Now

$$\cos(n_z, z-\zeta) = \frac{F(x) - F(\xi) + (\xi-x)F'(x)}{\{(x-\xi)^2 + (F(x) - F(\xi))^2\}^{\frac{1}{2}}\{1 + (F'(x))^2\}^{\frac{1}{2}}},$$

whence

$$\lim_{\zeta\to z} \frac{2\cos(n_z,z-\zeta)}{|z-\zeta|} =$$

$$\lim_{\xi\to x} \frac{2\{F(x) - F(\xi) + (\xi-x)F'(x)\}}{\{(x-\xi)^2 + (F(x) - F(\xi))^2\}\{1 + (F'(x))^2\}^{\frac{1}{2}}}.$$

Since

$$F(x) - F(\xi) + (\xi-x)F'(x) = (\xi-x)^2 \int_0^1 t\, F''(x + (\xi-x)t)dt$$

and

$$F(x) - F(\xi) = (x-\xi) \int_0^1 F'(x + (\xi-x)t)dt$$

we obtain

$$\lim_{\zeta\to z} \frac{2\cos(n_z,z-\zeta)}{|z-\zeta|} = F''(x)/\{1 + (F'(x))^2\}^{3/2},$$

which is the definition of the curvature κ. Since $F \in C^{k,\alpha}(\overline{D}_{\varepsilon,P})$ it follows that κ has continuous derivatives up to order $k-2$. □

In the two-dimensional case the potential due to a doublet distribution μ along the boundary is given by:

$$\phi_d(\zeta) = \frac{1}{2\pi} \oint_S \mu(z) \frac{\cos(n_z,z-\zeta)}{|z-\zeta|}\, dS_z, \tag{2.2}$$

with $\zeta \notin S$. The contour integration is taken along the boundary in a counter-clockwise direction. Since $\phi_d = \phi_d^{(2)}$ we further omit the upper index (2).

LEMMA 2.1. *Let* $S \in L^{2,\alpha}$ *and* $\mu \in H^{1,\alpha}(S)$. *If* ζ *approaches* S *we have* (Plemelj-Privalov formulae):

$$\phi_d^+(\zeta) = -\tfrac{1}{2}\mu(\zeta) - \tfrac{1}{2}\overline{\phi}_d(\zeta), \tag{2.2.1}$$

$$\phi_d^-(\zeta) = \tfrac{1}{2}\mu(\zeta) - \tfrac{1}{2}\overline{\phi}_d(\zeta), \tag{2.2.2}$$

with

$$\overline{\phi}_d(\zeta) = -\frac{1}{\pi}\oint_S \mu(z)\,\frac{\cos(n_z, z-\zeta)}{|z-\zeta|}\,ds_z, \quad \zeta \in S, \tag{2.2.3}$$

where ϕ_d^+ *and* ϕ_d^- *denote the limit from the outer and inner side respectively.*

PROOF. See MUSCHELISCHWILI [4, pp. 36-42, p.52]. □

For $z = \zeta$ the integrand in (2.2.3) is defined by its limit value, i.e. the curvature at ζ. In lemma 2.1 we assume that $S \in L^{2,\alpha}$ and taking into account remark 2.1 we conclude that $\kappa \in H^{0,\alpha}(S)$. Hence in (2.2.3) we may include the point ζ in the contour integration. Thus the integral in (2.2.3) may be interpreted as a proper integral.

The main result of this section is theorem 2.2. The proof of this theorem leans strongly on 3-D results given by GÜNTER [1, p.312], who has proved the following theorem: *let* $S \in L^{2,\alpha}$ *and* $\mu \in H^{0,\alpha}(S)$, *then* $\overline{\phi}_d^{(3)} \in H^{1,\alpha}(S)$. The reason why we cannot quote this theorem is that we assume μ to be essentially bounded, i.e. $\mu \in L_\infty(S)$.

DEFINITION 2.5. $L_\infty(S)$ denotes the space of essentially bounded functions on S. The associated norm is

$$\|\mu\|_\infty = \operatorname*{vrai\,max}_{z \in S} |\mu(z)|.$$

THEOREM 2.2. *Let* $S \in L^{2,\alpha}$ *and* $\mu \in L_\infty(S)$, *then*

$$\overline{\phi}_d \in H^{1,\alpha}(S).$$

PROOF. Let (ξ,η) be a local coordinate system about a certain point $P \in S$. Using definition 2.3 we split the boundary into two parts Σ and $S-\Sigma$. Let $\zeta \in \Sigma$. For (2.2.3) we obtain

$$\overline{\phi}_d(\zeta) = -\frac{1}{\pi}\int_{S-\Sigma} \mu(z)\,\frac{\cos(n_z,z-\zeta)}{|z-\zeta|}\,dS_z - \frac{1}{\pi}\int_{\Sigma} \mu(z)\,\frac{\cos(n_z,z-\zeta)}{|z-\zeta|}\,dS_z$$

In the first integral $z \in S-\Sigma$ and therefore $|z-\zeta| \neq 0$. If we replace ζ by $(\xi,F(\xi))$ we obtain a function of ξ which has bounded and continuous derivatives up to order 2 (since $S \in L^{2,\alpha}$); hence the first integral certainly belongs to the class $H^{1,\alpha}(S)$. We proceed to establish that the second integral also belongs to $H^{1,\alpha}(S)$. We denote the coordinates of the point ζ by ξ,η and those of the integration point z by x,y. Substituting $\eta = F(\xi)$ and $y = F(x)$ we obtain

$$\begin{aligned} I_2 &\equiv \int_{\Sigma} \mu(z)\,\frac{\cos(n_z,z-\zeta)}{|z-\zeta|}\,dS_z \\ &= \int_{\overline{D}_{\varepsilon,P}} \mu(x)\,\frac{F(x)-F(\xi)+(\xi-x)F'(x)}{\{(x-\xi)^2+(F(x)-F(\xi))^2\}\{1+(F'(x))^2\}^{\frac{1}{2}}}\,dx. \end{aligned}$$

We define the following functions

$$\psi_1(\xi,x) = \int_0^1 F'(\xi+(x-\xi)t)\,dt$$

and

$$\psi_2(\xi,x) = \int_0^1 tF''(\xi+(x-\xi)t)\,dt.$$

Integrating by parts, we obtain:

$$F(x) - F(\xi) + (\xi-x)F'(x) = -(\xi-x)^2\,\psi_2(\xi,x), \tag{2.3}$$

and

(2.4) $$F(x) - F(\xi) = (x-\xi)\,\psi_1\,(\xi,x).$$

Hence, the second integral becomes

(2.5) $$I_2 = -\int_{\overline{D}_{\varepsilon,P}} \mu(x) \frac{\psi_2(\xi,x)}{(1+\psi_1^2(\xi,x))(1+(F'(x))^2)^{\frac{1}{2}}}\,dx.$$

Assuming $\|\mu\|_\infty < A$ we have to prove that:

(2.6) $$\left|\frac{dI_2}{d\xi}\right| < C\,A,$$

and

(2.7) $$\left|\frac{d}{d\xi} I_2(\xi_1) - \frac{d}{d\xi} I_2(\xi_2)\right| < C\,A\,|\xi_1-\xi_2|^\alpha, \quad \forall \xi_1,\xi_2 \in \overline{D}_{\varepsilon,P}.$$

First, we show that

$$|\psi_2(\xi_1,x) - \psi_2(\xi_2,x)| \le C\,|\xi_1-\xi_2|^\alpha.$$

Indeed, since $F'' \in C^{0,\alpha}(\overline{D}_{\varepsilon,P})$, we have:

$$\begin{aligned}
|\psi_2(\xi_1,x) - \psi_2(\xi_2,x)| &= \left|\int_0^1 t\{F''(x+(\xi_1-x)t) - F''(x+(\xi_2-x)t)\}dt\right| \\
&\le \int_0^1 |F''(x+(\xi_1-x)t) - F''(x+(\xi_2-x)t)|dt \\
&\le \int_0^1 |x+(\xi_1-x)t-x+(\xi_2-x)t|^\alpha\,dt \\
&\le C\,|\xi_1-\xi_2|^\alpha.
\end{aligned}$$

In order to prove (2.6) and (2.7) we first investigate the function

$$\frac{\psi_2(\xi,x)}{1+\psi_1^2(\xi,x)} \quad . \tag{2.8}$$

We denote this function by R. Since ψ_1 and ψ_2 are bounded, it follows that R is bounded too. We shall now prove the following inequalities:

$$\left|\frac{\partial R}{\partial \xi}\right| < \frac{C}{|\xi-x|^{1-\alpha}} \qquad \text{for } \xi \to x, \tag{2.9}$$

$$\left|\frac{\partial^2 R}{\partial \xi^2}\right| < \frac{C}{|\xi-x|^{2-\alpha}} \qquad \text{for } \xi \to x. \tag{2.10}$$

Differentiating R we obtain:

$$\frac{\partial R}{\partial \xi} = \frac{\partial \psi_2}{\partial \xi}(1+\psi_1^2)^{-1} - 2(1+\psi_1^2)^{-2}\psi_1 \frac{\partial \psi_1}{\partial \xi}\psi_2. \tag{2.11}$$

It can be easily proved that $\left|\frac{\partial \psi_1}{\partial \xi}\right| < c_0$. Since $|1+\psi_1^2| > 1$, $|\psi_1| < c_1$ and $|\psi_2| < c_2$ (with c_0, c_1, c_2 certain constants that depend on F) it follows that

$$\left|\frac{\partial R}{\partial \xi}\right| < \left|\frac{\partial \psi_2}{\partial \xi}\right| + 2c_0c_1c_2.$$

Inequality (2.9) will have been proved when we have shown that

$$\left|\frac{\partial \psi_2}{\partial \xi}\right| < \frac{C}{|\xi-x|^{1-\alpha}} \quad . \tag{2.12}$$

From (2.3) we obtain

$$\frac{\partial \psi_2}{\partial \xi} = 2\,\frac{\Psi_2}{(x-\xi)} + \frac{F'(\xi)-F'(x)}{(x-\xi)^2} \quad . \tag{2.13}$$

Since

$$F'(\xi) - F'(x) = -(x-\xi)\int_0^1 F''(\xi+(x-\xi)t)\,dt,$$

it follows that

$$\frac{\partial\psi_2}{\partial\xi} = \{2\int_0^1 tF''(\xi+ (x-\xi)t)dt - \int_0^1 F''(\xi+ (x-\xi)t)dt\}/(x-\xi)$$

$$= \frac{1}{(x-\xi)} \int_0^1 \{F''(\xi+ (x-\xi)\sqrt{t}) - F''(\xi + (x-\xi)t)\}dt$$

Hence

$$\left|\frac{\delta\psi_2}{\delta\xi}\right| \le \frac{C}{|x-\xi|} \int_0^1 |\xi+ (x-\xi)\sqrt{t} - \xi - (x-\xi)t|^\alpha \, dt$$

$$\le \frac{C}{|x-\xi|^{1-\alpha}} \int_0^1 |\sqrt{t}- t|^\alpha \, dt \le \frac{c}{|x-\xi|^{1-\alpha}} .$$

Using (2.9) and (2.12) we obtain by another differentiation of R the estimate:

$$\left|\frac{\partial^2 R}{\partial\xi^2}\right| < \left|\frac{\partial^2\psi_2}{\partial\xi^2}\right| + \frac{c_3}{|x-\xi|^{1-\alpha}} + c_4 .$$

Therefore it suffices to show that

$$\left|\frac{\partial^2\psi_2}{\partial\xi^2}\right| < \frac{c}{|x-\xi|^{2-\alpha}} .$$

Differentiating (2.13) we obtain:

$$\frac{\partial^2\psi_2}{\partial\xi^2} = \frac{2\psi_2}{(x-\xi)^2} + \frac{2}{(x-\xi)} \frac{\partial\psi_2}{\partial\xi} + \frac{2}{(x-\xi)^3} \{F'(\xi) - F'(x)\} + \frac{F''(\xi)}{(x-\xi)^2}$$

$$= \frac{3}{(x-\xi)} \frac{\partial\psi_2}{\partial\xi} + \frac{1}{(x-\xi)^3} \{F'(\xi) - F'(x)\} + \frac{F''(\xi)}{(x-\xi)^2}$$

$$= \frac{3}{(x-\xi)} \frac{\partial\psi_2}{\partial\xi} + \frac{1}{(x-\xi)^2} \left[F''(\xi) - \int_0^1 F''(\xi+ (x-\xi)t)dt\right] .$$

Because of the mean value theorem the expression within square brackets is equal to

$$F''(\xi) - F''(\xi + (x-\xi)t^*), \text{ for some } t^* \in [0,1].$$

Since $F'' \in C^{0,\alpha}(\overline{D}_{\varepsilon,P})$ and $\left|\frac{\partial\psi_2}{\partial\xi}\right| < \frac{c}{|x-\xi|^{1-\alpha}}$ it follows that

$$\left|\frac{\partial^2\psi_2}{\partial\xi^2}\right| < \frac{c}{|x-\xi|^{2-\alpha}} .$$

We now consider the integral

$$\frac{dI_2}{d\xi}(\xi) = -\int_{\overline{D}_{\varepsilon,P}} \mu(x)\, \frac{\partial R}{\partial\xi}(\xi,x)\, \frac{1}{(1+(F'(x))^2)^{\frac{1}{2}}}\, dx.$$

Without loss of generality we can take $\overline{D}_{\varepsilon,P}$ equal to $[0,1]$. Since $\|\mu\|_\infty \le A$ and $|1 + F'^2(x)| \ge 1$ it follows that

$$\left|\frac{dI_2}{d\xi}\right| \le A \int_0^1 \left|\frac{\partial R}{\partial\xi}(\xi,x)\right| dx.$$

Using estimate (2.9) we conclude that the singularity in $\frac{\partial R}{\partial\xi}$ is integrable. Hence, $\frac{dI_2}{d\xi}$ is bounded. We proceed to establish (2.7). Ket $\delta = |\xi_1-\xi_2|$, then

$$(2.14)\qquad \left|\frac{dI_2}{d\xi}(\xi_1) - \frac{dI_2}{d\xi}(\xi_2)\right| \le A \int_0^1 \left|\frac{\partial R}{\partial\xi}(\xi_1,x) - \frac{\partial R}{\partial\xi}(\xi_2,x)\right| dx$$

$$\le A \int_{\xi_1-2\delta}^{\xi_1+2\delta} \left|\frac{\partial R}{\partial\xi}(\xi_2,x)\right| dx + A \int_{\xi_1-2\delta}^{\xi_1+2\delta} \left|\frac{\partial R}{\partial\xi}(\xi_1,x)\right| dx +$$

$$A \int_0^{\xi_1-2\delta} \left|\frac{\partial R}{\partial\xi}(\xi_1,x) - \frac{\partial R}{\partial\xi}(\xi_2,x)\right| dx + A \int_{\xi_1+2\delta}^1 \left|\frac{\partial R}{\partial\xi}(\xi_1,x) - \frac{\partial R}{\partial\xi}(\xi_2,x)\right| dx.$$

Because of inequality (2.9) we obtain for the second integral on the right-hand side of (2.14):

$$\int_{\xi_1-2\delta}^{\xi_1+2\delta} \left|\frac{\partial R}{\delta\xi}(\xi_1)\right| dx < \int_{\xi_1-2\delta}^{\xi_1+2\delta} |\xi_1-x|^{\alpha-1} dx < c(2\delta)^{\alpha}.$$

Since the interval $|\xi_1-x| < 2\delta$ is contained in the circle $|\xi_2-x| < 3\delta$ we obtain for the first integral the estimate $c(3\delta)^{\alpha}$. Therefore the sum of the first two integrals is less then a number of the form $c\delta^{\alpha}$.

Finally, we have to estimate the last two integrals of (2.14). For $x \notin [\xi_1,\xi_2]$ the mean value theorem yields

$$\frac{\partial R}{\partial\xi}(\xi_1,x) - \frac{\partial R}{\partial\xi}(\xi_2,x) = (\xi_1-\xi_2)\,\frac{\partial^2 R}{\partial\xi^2}(\xi^*,x),$$

where ξ^* denotes some point of the interval $[\xi_1,\xi_2]$. From inequality (2.10)

$$\left|\frac{\partial R}{\partial\xi}(\xi_1,x) - \frac{\partial R}{\partial\xi}(\xi_2,x)\right| < \delta c|\xi^*-x|^{\alpha-2},$$

so that for the third integral the following estimate is obtained:

$$c\delta \int_0^{\xi_1-2\delta} (\xi^*-x)^{\alpha-2} dx = \frac{c\delta}{\alpha-1}\left[(\xi^*-\xi_1+2\delta)^{\alpha-1} - \xi^{*\alpha-1}\right] < \bar{c}\,\delta^{\alpha}.$$

In the same way we obtain a similar estimate for the last integral. Hence, the left-hand side of (2.14) is less than a number of the form $c\,\delta^{\alpha} A$. By definition $\delta = |\xi_1-\xi_2|$ and thus inequality (2.7) has been proved. This completes the proof of theorem 2.2. □

One of the most important applications of potential theory is solving Dirichlet and Neumann problems. The solution of a Dirichlet problem can be sought in the form of the double-layer potential ϕ_d (2.2). In the case of an interior Dirichlet-problem the boundary value ϕ_d^- is prescribed and the solution ϕ_d follows from (2.2) as soon as the doublet distribution μ has been determined from equation (2.2.2). Let the integral operator in (2.2.3) be denoted symbolically by K:

$$(2.15) \qquad K\mu(\zeta) = \frac{-1}{\pi} \oint_S \mu(z) \frac{\cos(n_z, z-\zeta)}{|z-\zeta|} dS_z, \quad \zeta \in S.$$

From theorem 2.2 it follows that $K\mu \in H^{1,\alpha}(S)$ if $S \in L^{2,\alpha}$ and $\mu \in L_\infty(S)$. We conclude that the operator K maps from the Banach space $L_\infty(S)$ into the class $H^{1,\alpha}(S)$, which is a Banach space too if it is equipped with the following norm:

$$\|f\|_{1,\alpha} = \sum_{\ell=0}^{1} \sum_{i=0}^{\ell} \left\|\frac{\partial^i f}{\partial t^i}\right\| + \sum_{\ell=0}^{1} \left\|\frac{\partial^\ell f}{\partial t^\ell}\right\|_\alpha,$$

where $\frac{\partial}{\partial t}$ denotes differentiation in the tangential direction and

$$\|f\|_\alpha = \|f\| + \sup_{z_1,z_2 \in S} \frac{|f(z_1) - f(z_2)|}{|z_1-z_2|^\alpha}.$$

Since K is a linear operator corollary 2.3 follows directly from theorem 2.2:

COROLLARY 2.3. *Let* $S \in L^{2,\alpha}$, *then the operator* K *mapping from* $L_\infty(S)$ *into* $H^{1,\alpha}(S)$ *is bounded*, i.e.:

$$\|K\mu\|_{1,\alpha} \le C \|\mu\|_\infty, \quad \text{for all } \mu \in L_\infty(S).$$

We note that the space $H^{1,\alpha}(S)$ is compactly imbedded in the space $L_\infty(S)$. From this property and the previous corollary follow:

COROLLARY 2.4. *Let* $S \in L^{2,\alpha}$, *then the operator* K *mapping from* $L_\infty(S)$ *into* $L_\infty(S)$ *is compact.*

REMARK 2.2. From the Fredholm alternative theorem for compact operators it follows that equation (2.2.2) has a unique solution for each boundary function $\phi_d^- \in L_\infty(S)$ if $S \in L^{2,\alpha}$ (see also ZABREYKO [8, p.218]). In addition, the operator $(I-K)^{-1}$ is bounded on the space $L_\infty(S)$,

$$\|(I-K)^{-1}\|_\infty < C.$$

COROLLARY 2.5. *Let* $S \in L^{2,\alpha}$ *and* $\phi_d^- \in H^{1,\alpha}(S)$, *then the solution of* (2.2.2) *belongs to* $H^{1,\alpha}(S)$.

PROOF. From remark 2.2 we have $\mu \in L_\infty(S)$. But (2.2.2) can be written as

$$\mu = 2\phi_d^- + K\mu.$$

By theorem 2.2 it follows that the right-hand side belongs to $H^{1,\alpha}(S)$. □

3. AERODYNAMIC APPLICATION

For incompressible and irrotational flow around a two-dimensional body there exists a velocity potential ϕ satisfying Laplace's equation

$$(3.1) \qquad \Delta\phi = 0,$$

with boundary conditions

$$(3.2) \qquad \frac{\partial\phi}{\partial n_\ell} = 0 \qquad \text{along the boundary } S,$$

and

$$(3.3) \qquad \phi(\zeta) \to \mathcal{U} \,.\, \zeta \qquad \text{for } |\zeta| \to \infty,$$

where $\frac{\partial}{\partial n_\ell}$ denotes differentiation in the direction of the outward normal to S and $\mathcal{U}.\zeta$ denotes the real part of the complex number $\mathcal{U}\zeta$. The velocity potential ϕ is given by the superposition

$$(3.4) \qquad \phi(\zeta) = \phi_d(\zeta) + \mathcal{U} \,.\, \zeta,$$

where ϕ_d is defined by (2.2) and μ the solution of equation (3.6.1). By standard arguments the above Neumann problem for the exterior of the boundary S is transformed to a Dirichlet problem in the interior.

LEMMA 3.1. *Let* $S \in L^{2,\alpha}$ *and* $\mu \in H^{1,\alpha}(S)$, *then the boundary condition* (3.2) *can be replaced by:*

$$(3.5) \qquad \phi^-(\zeta) = 0, \qquad \zeta \in S.$$

PROOF. See Martensen [3, p.247]. □

From lemma 3.1 and the equations (3.4) and (3.5) it follows that condition (3.2) is satisfied if the doublet distribution μ is the solution of the following Fredholm integral equation:

$$(3.6.1) \qquad \mu(\zeta) + \frac{1}{\pi} \oint_S \mu(z) \frac{\cos(n_z, z-\zeta)}{|z-\zeta|} dS_z = -2\mathcal{U}.\zeta.$$

We write this equation in operator notation as

$$(3.6.2) \qquad (I - K)\mu = g,$$

where $g(\zeta) = -2\mathcal{U}.\zeta$ and K the integral operator defined by (2.15). From remark 2.2 it follows that $(I - K)$ has a bounded inverse on $L_\infty(S)$. Further it can be verified that $g \in H^{1,\alpha}(S)$ and as a consequence of corollary 2.5 we obtain $\mu \in H^{1,\alpha}(S)$.

Now we discuss the convergence of a sequence of approximations to the unique solution of (3.6.2). We divide the boundary S into N segments S_i, so that $S = S_1 + S_2 + \dots + S_N$. The begin- and end-points of the i^{th} segment are z_{i-1} and z_i, which are called nodal points. We approximate the function $\mu(\zeta)$, $\zeta \in S$, by a step-function and we solve the resulting equation by a collocation method. The collocation points $\zeta_i, i = 1,2,\dots,N$, are taken to be the mid-points of the segments S_i.

DEFINITION 3.1. Let (x,y) be a local coordinate system about a certain point $P \in S_i$ and let the coordinates of the nodal point z_i be given by $(x_i, F(x_i))$ with $F(x)$ as in definition 2.3. The coordinates of the collocation point ζ_i are defined by $(\xi, F(\xi))$ with $\xi = (x_{i-1} + x_i)/2$.

We write the approximating step-function in operator notation as follows:

$$(3.7) \qquad T_N\mu(\zeta) = \sum_{i=1}^{N} \mu(\zeta_i)\, u_i(\zeta)$$

with

$$u_i(\zeta) = \begin{cases} 1, & \zeta \in S_i, \\ 0, & \zeta \notin S_i. \end{cases}$$

We define T_N as a linear mapping from the space $C(S)$ of continuous functions on S (with the supremum norm $\|\cdot\|$) to $X_N = \mathrm{span}(u_1,\ldots,u_N)$. It is noteworthy to remark that T_N is not a projection operator in $C(S)$, because X_N is not a subspace of $C(S)$. However, $X_N \subset L_\infty(S)$ and it is easy to prove the following.

<u>LEMMA 3.2</u>. *The mappings* T_N *and* $I - T_N$ *are bounded from* $C(S)$ *into* $L_\infty(S)$.

<u>PROOF</u>. For all $f \in C(S)$ we have

$$\sup_{f\in C(S)} \|T_N f\|_\infty = \max_{1\le i\le N} |f(\zeta_i)| \le \|f\|. \quad \square$$

Let h_N be a measure of the mesh-size defined by:

$$h_N = \max_{1\le i\le N} |z_i - z_{i-1}| .$$

We assume that the partition of the boundary is such that $\lim_{N\to\infty} h_N = 0$.

<u>LEMMA 3.3</u>. *Let* $S \in L^{2,\alpha}$ *and* $f \in H^{1,\alpha}(S)$, *then*

$$\|(I - T_N)f\|_\infty \le C\, h_N \|f\|_{1,\alpha} \quad \text{as } N\to\infty.$$

<u>PROOF</u>. Draw a circle with centre ζ_i and radius h_N. The proof follows from definition 2.4. $\square$

For a given N an approximate solution of equation (3.6.2) is obtained by solving:

$$(3.8) \qquad (I - T_N K)\mu_N = T_N g, \quad \mu_N \in X_N.$$

REMARK 3.1. In aerodynamics this collocation method has become very popular because $T_N K\mu_N$ can be easily calculated. In the two-dimensional case angles have to be measured.

REMARK 3.2. As a consequence of theorem 2.2 an approximation in the space $H^{1,\alpha}(S)$ can be obtained by a single iteration

$$\tilde{\mu}_N = g + K\mu_N, \tag{3.9}$$

where μ_N is the solution of (3.8). It is easily verified that

$$\tilde{\mu}_N = g + KT_N\tilde{\mu}_N.$$

For the case that T_N represents a projection operator and K a sufficiently regular integral operator the convergence properties of $\tilde{\mu}_N$ are discussed by SLOAN [7].

REMARK 3.3. Real aerofoils are given by a data-set of points $\{x_i,y_i\}_{i=1}^{M}$. Usually a continuous boundary is obtained by a polygon connecting the points of the data-set. However, this polygon does not belong to the class $L^{2,\alpha}$ and a single iteration does not yield an approximation in the space $H^{1,\alpha}(S)$. Therefore, if one wants the approximate solution $\tilde{\mu}_N$ in $H^{1,\alpha}(S)$, it is necessary to construct a smoother boundary through the points $\{x_i,y_i\}$, e.g. a cubic spline approximation so that $S \in L^{2,\alpha}$, except for a small region near the trailing edge.

LEMMA 3.4. *Let the finite-dimensional subspace* $X_N \subset L_\infty(S)$ *be sufficiently large (i.e. the mesh-size of the discretization is sufficiently small) and let* $S \in L^{2,\alpha}$. *From the existence of a bounded inverse of* $I-K$ *on* $L_\infty(S)$ *follow:*

$$(I - T_N K)^{-1} \text{ exists on } L_\infty(S)$$

and

$$C_1 = \sup_{n\geq N} \|(I - T_n K)^{-1}\|_{L_\infty(S)\to L_\infty(S)} < \infty.$$

PROOF. For each $f \in L_\infty(S)$ we have by Lemma 3.3 and Corollary 2.3:

$$\| (I - T_N)Kf\|_\infty \le c_2 h_N \|Kf\|_{1,\alpha} \le c_3 h_N \|f\|_\infty .$$

But then $\|K-T_N K\|_{L_\infty(S) \to L_\infty(S)} \to 0$ as $N \to \infty$ and existence and boundedness on $L_\infty(S)$ follow from Neumann's theorem. See also PRENTER [5, p.574].

LEMMA 3.5. *Let* $S \in L^{2,\alpha}$ *and* $f \in H^{1,\alpha}(S)$, *then*

$$\left| \int_{S_i} \{f(z) - f(\zeta_i)\} dS_z \right| \le C_2 h_N^{2+\alpha} \|f\|_{1,\alpha}, \text{ as } N \to \infty.$$

PROOF. Let (x,y) be a local coordinate system about a certain point $P \in S_i$. We denote the coordinates of the point ζ_i by (ξ,η) and those of the integration point z by (x,y). Using definition 2.4 we can represent $f(z) - f(\zeta_i)$ by

$$\hat{f}(x) - \hat{f}(\xi) = (x-\xi)\hat{f}'(\xi) + (x-\xi) \int_0^1 \{\hat{f}'(\xi + (x-\xi)t) - \hat{f}'(\xi)\} dt.$$

We recall that ζ_i is the midpoint of S_i. Following definition 3.1 we denote the coordinates of the nodal-point z_i by (x_i, y_i). Letting $h = (x_i - x_{i-1})/2$ the above integral can be estimated as follows:

$$\left| \int_{S_i} \{f(z) - f(\zeta_i)\} dS_z \right| \le \left| \int_{\xi-h}^{\xi+h} \{\hat{f}(x) - \hat{f}(\xi)\} dx \right| \le$$

$$\left| \int_{\xi-h}^{\xi+h} (x-\xi)\hat{f}'(\xi) dx \right| + \left| \int_{\xi-h}^{\xi+h} (x-\xi) \int_0^1 \{\hat{f}'(\xi + (x-\xi)t) - \hat{f}'(\xi)\} dt\, dx \right|$$

The first part is equal to zero. We proceed to estimate the second part. Letting $x - \xi = v$ we have

$$\begin{aligned} I_2 &\le \left| \int_{-h}^{h} v \int_0^1 \{\hat{f}'(\xi+vt) - \hat{f}'(\xi)\} dt\, dv \right| \\ &\le \int_{-h}^{h} |v| \int_0^1 |\hat{f}'(\xi+vt) - \hat{f}'(\xi)| dt\, dv \\ &\le 2 \int_0^h |v|^{1+\alpha} \|f\|_{1,\alpha}\, dv \le C_2 h_N^{2+\alpha} \|f\|_{1,\alpha}. \quad \square \end{aligned}$$

In theorem 3.6 we discuss the convergence of the approximate solutions $\mu_N \in X_N$ and $\tilde{\mu}_N \in H^{1,\alpha}(S)$ to the exact solution μ. We give error estimates for $\|\mu-\mu_N\|_\infty$, $\|T_N\mu-\mu_N\|_\infty$ and $\|\mu-\tilde{\mu}_N\|$.

<u>THEOREM 3.6</u>. [*Approximation theorem*]. *Let the boundary* $S \in L^{2,\alpha}$, *then for* $N \to \infty$:

(i) $\|\mu-\mu_N\|_\infty \le C_3 h_N \|\mu\|_{1,\alpha}$, *where* μ *is the solution of* (3.6.2) *and* μ_N *of* (3.8).

(ii) $\|K(I - T_N)f\| \le C_4 h_N^{1+\alpha} \|f\|_{1,\alpha}$, *for all* $f \in H^{1,\alpha}(S)$.

(iii) $\|T_N\mu - \mu_N\|_\infty \le C_5 h_N^{1+\alpha} \|\mu\|_{1,\alpha}$,

(iv) $\|\mu - \tilde{\mu}_N\| \le C_6 h_N^{1+\alpha} \|\mu\|_{1,\alpha}$.

<u>PROOF</u>.

(i) From (3.6.2) and (3.8) we get:

$$(I - T_N K)(\mu - \mu_N) = \mu - T_N K\mu - T_N g = \mu - T_N\mu.$$

Use lemma 3.4 and 3.3 to obtain:

$$\|\mu-\mu_N\|_\infty \le C_1\|\mu-T_N\mu\|_\infty \le C_3 h_N \|\mu\|_{1,\alpha}.$$

(ii) From the construction of $T_N f$ it follows that

$$K(I - T_N)f(\zeta) = \sum_{i=1}^{N} \int_{S_i} \{f(z) - f(\zeta_i)\} \frac{\cos(n_z, z-\zeta)}{|z-\zeta|} dS_z.$$

Without of loss of generality we can take $\zeta \in S_i$. Taking into account remark 2.1 we estimate the (i)-th part of the above sum by:

$$\left| \int_{S_i} \{f(z) - f(\zeta_i)\} \frac{\cos(n_z, z-\zeta)}{|z-\zeta|} dS_z \right| \le$$

$$C \int_{S_i} |f(z) - f(\zeta_i)| dS_z \le C' \int_{\xi-h}^{\xi+h} |x| \, \|f\|_{1,\alpha} dx \le C'' h_N^2 \|f\|_{1,\alpha}$$

where ξ, h are defined in the proof of Lemma 3.5.
In the other parts of the above sum we replace z by $(x,F(x))$. Since $S \in L^{2,\alpha}$ the kernel-function $\cos(n_z, z-\zeta)/|z-\zeta|$ has bounded and continuous derivatives up to order 2 with respect to x. Hence this function can be written as a series expansion involving powers of $(x-\xi)$. Applying Lemma 3.5 we obtain

$$\|K(I - T_N)f\| \leq C'' h_N^2 \|f\|_{1,\alpha} + (N-1)C_2 h_N^{2+\alpha} \|f\|_{1,\alpha}$$

$$\leq C_4 h_N^{1+\alpha} \|f\|_{1,\alpha}.$$

(iii) From equation (3.6.2) we get

$$(I - T_N K)T_N\mu = T_N g + T_N K(\mu - T_N\mu)$$

and subtract (3.8) to obtain

$$(I - T_N K)(T_N\mu - \mu_N) = T_N K(\mu - T_N\mu).$$

Applying Lemma 3.4 and 3.2 we have

$$\|T_N\mu - \mu_N\|_\infty \leq C \|K(\mu - T_N\mu)\|.$$

Since $\mu \in H^{1,\alpha}(S)$ the proof follows from part (ii) of this theorem.

(iv) From (3.6.2) and (3.9) it follows that

$$\|\mu - \tilde{\mu}_N\| = \|K(\mu - \mu_N)\|$$

$$\leq \|K(\mu - T_N\mu)\| + \|K(T_N\mu - \mu_N)\|.$$

Using part (ii) and (iii) we obtain the proof of (iv). □

With respect to the smoothness of the boundary S, part *(ii)* of theorem 3.6 is a modification of results given by KANTOROWITSCH [2, p.127]. He has proved the following: let the boundary S be given by the parametric equations

$$z(t) = X(t) + iY(t), \quad t \in [0,1],$$

and let $\omega(s,t) = \arg(z(s) - z(t))$. If ω is three times continuously differentiable with respect to s (this assumption is stronger than $S \in L^{2,\alpha}$!) and the function f is two times continuously differentiable (i.e. $f \in C^{(2)}[0,1]$) then

$$\|K(I - T_N)f\|_\infty \in C\, h_N^2 \, \|f\|_2,$$

where $\|\cdot\|_2$ is the usual norm of the space $C^{(2)}[0,1]$.

Usually part *(iii)* of theorem 3.6 is called super-convergence on the collocation points. Performing a single iteration of type (3.9) the order of super-convergence is extended to all points of the boundary as has been shown by part *(iv)*.

So far we did not say anything about how to solve equation (3.8). When the dimension of X_N is small it can be solved by a direct method (e.g. Gaussian elimination). However, when the dimension is large one usually uses iterative techniques. In [6] we have applied a multiple grid iterative proces to (3.8) and we have estimated its reduction factor η by

$$\eta = \|(I - T_N)K\|_{X_N \to X_N} .$$

Using Corollary 2.3 and Lemma 3.3 we obtain that $\eta < C\, h_N$, as $N \to \infty$. Indeed, the reduction factor decreases as N increases. Then, asymptotically for $N \to \infty$, the multiple grid method needs only 2 iterations to obtain a result for which the superconvergence on the collocation points is preserved.

We have applied the numerical method of this section to the calculation of non-circulatory, potential flow around a Kármán-Trefftz aerofoil. This aerofoil is obtained from the circle in the χ-plane, $\chi = c\, e^{i\theta}$, by means of the mapping

$$(3.10) \qquad z = f(\chi) = (\chi-\chi_t)^k/(\chi-c(\delta-i\gamma))^{k-1},$$

where k measures the trailing edge angle, γ the camber and δ the thickness of the aerofoil;

$$c = 2\ell(\delta + \sqrt{1-\gamma^2})^{k-1} / (2\sqrt{1-\gamma^2})^k$$

$$\chi_t = c(\sqrt{1-\gamma^2} - i\gamma),$$

with ℓ the length of the aerofoil. To make f single-valued we take the principal value in (3.10).

The Kármán-Trefftz aerofoil does not belong to the class $L^{2,\alpha}$ because of the presence of the trailing edge at $z = z_t$. At this point the curvature is not defined. In the present paper we remove the corner by the additional mapping

$$w = g(z) = z(1 - \tilde{z}/z)^{1-1/k}, \tag{3.11}$$

where $\tilde{z}$ is a point inside the aerofoil. Here we locate it arbitrarily at $\tilde{z} = -1.95$. By means of (3.11) the aerofoil in the z-plane is converted into a quasi-circular shape in the w-plane. This has been done because the inverse mapping of (3.10) converts real aerofoils (which do not belong to the family of Kármán-Trefftz aerofoils) into quasi-circular shapes too.

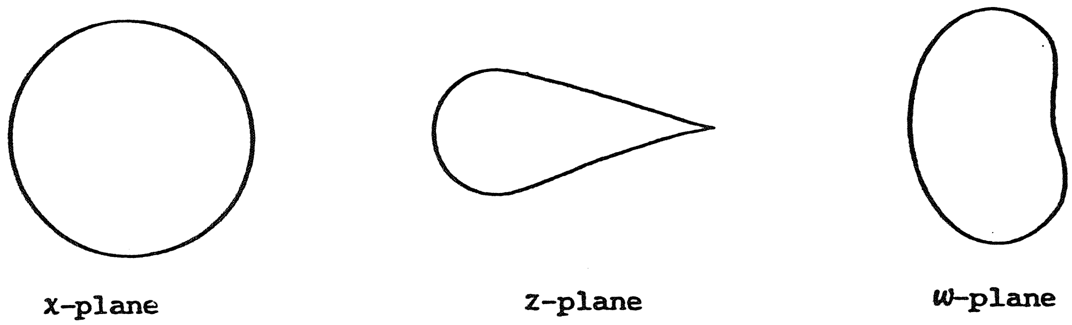

figure 3.1

We divide the circle in the χ-plane uniformly into N segments. Hence the nodal-points in this plane are given by:

$$\chi_j = ce^{i\theta_j}, \quad \theta_j = 2\pi j/N, \quad j = 0,1,\ldots,N.$$

Substituting x_j, $j = 0,1,\dots,N$, into (3.10) and (3.11) successively, we obtain the nodal-points $\{z_j\}$ in the z-plane and $\{\omega_j\}$ in the ω-plane.

The tangential velocity V_j at the point z_j ($z_j \in$ K.T.-aerofoil) is obtained numerically by:

$$V_j = \frac{|\mu_{N,j+1} - \mu_{N,j}|}{|\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}|} * \left|\frac{dw}{dz}\right|_{z=z_j}, \quad j = 1,\dots,N-1,$$

where $\omega_{j+\frac{1}{2}}$ is the collocation point corresponding with $x_{j+\frac{1}{2}}$. We note that V_j is obtained by numerical differentiation. Therefore, from theorem 3.6 (*iv*) we obtain the following error estimate

$$\text{(3.12)} \qquad \max_{1\le j\le N-1} |V_j - V_{exact}(z_j)| \le ch_N^{\alpha}, \text{ as } N \to \infty.$$

We have taken the following test cases:

(a) $k = 1.90$, $\delta = 0.05$, $\ell = 1.0$, $\gamma = 0.0$, $U = 1.0$,

(b) $k = 1.99$, $\delta = 0.05$, $\ell = 1.0$, $\gamma = 0.0$, $U = 1.0$.

In table I we give the maximum error in the tangential velocity (i.e. the left-hand side of (3.12) for increasing values of N. For the above test cases the error estimate (3.12) is found to be too pessimistic.

N \ k	1.90	1.99
32	.12 (-1)	.53 (-1)
64	.62 (-2)	.26 (-1)
128	.16 (-2)	.66 (-2)
256	.39 (-3)	.19 (-2)

Table I. Maximum error in tangential velocity
($\delta = 0.05$, $\ell = 1.0$, $\gamma = 0.0$, $U = 1.0$)

For N = 64, 128 and 256 the above results have been obtained by a multiple grid iterative proces.

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